Test functions

We shall now focus on $X = \mathbb{R}^n$ endowed w/ Lebesgue measure, m or dx. For open $U \subseteq \mathbb{R}^n$,

$$\mathcal{C}^k(U) = \left\{ f : U \to \mathbb{C} : f_{x^\alpha} = \partial^\alpha f = \frac{\partial^\alpha f}{\partial x^\alpha} \text{ exist and are cont. for all } |\alpha| \le k \right\}$$

Multi-index notation: $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\ge 0}$

$|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$, $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$

$z = (x_1, \dots, x_n)$ or $(\partial_1, \dots, \partial_n) \dots$

$$\mathcal{C}^\infty(U) = \bigcap_{k \ge 0} \mathcal{C}^k(U).$$

$$\operatorname{supp} f = \overline{\{x : f(x) \ne 0\}}.$$

$$\underbrace{\mathcal{C}_c^\infty(U)}_{\text{test fcns}} = \{ f \in \mathcal{C}^\infty(U) : \operatorname{supp} f \underset{\text{compact}}{\subseteq_c} U \}$$

Rem. $\mathcal{C}_c^\infty(U) \cong \{ f \in \mathcal{C}_c^\infty(\mathbb{R}^n) = \mathcal{C}_c^\infty :$
$\operatorname{supp} f \subset\subset U \}$.

Basic Lemma. $\exists\, \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $0 \le \varphi \le 1$
$\operatorname{supp}\varphi = \overline{B^n} = \{ x \in \mathbb{R}^n ;\ |x| \le 1 \}$.

$$|x|^2 = x \cdot x = \sum_{j=1}^n x_j^2 .$$

Pf. Start w/ $\psi : \mathbb{R} \to [0,1]$ given by

$$\psi(t) = \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \le 0 . \end{cases}$$

Clearly $\psi \in \mathcal{C}^\infty$ on $(-\infty,0) \cup (0,\infty)$, and $\psi(x) \to 0$ as $x \to 0^+$. To show that $\psi \in \mathcal{C}^\infty$, note that for any $k \in \mathbb{Z}_{\ge 0}$ and $t > 0$,

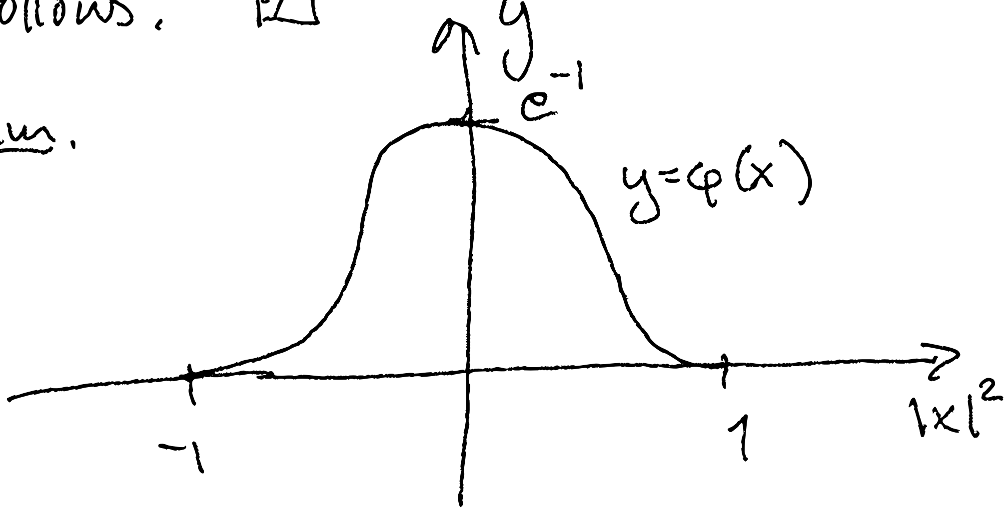
$$\psi(t) = p\left(\tfrac{1}{t}\right) e^{-1/t}, \quad p \text{ is polynomial.}$$

Since $\frac{1}{t^{\ell}} e^{-1/t} \to 0$ as $t \to 0^+$ for any ℓ, we conclude that $\psi^{(k)}(t) \to 0$ as $t \to 0^+ \Rightarrow \psi \in C^\infty(\mathbb{R})$. Also, $0 \le \psi \le 1$.

Now, set $\varphi(x) = \psi(1-|x|^2)$.

Since $1-|x|^2$ is C^∞, $1-|x|^2 \geq 0$ in $\overline{B^n}$, the conclusion of Basic Lemma follows. $\boxtimes$

<u>Rem.</u>



Another important class of test functions is the Schwartz space, $\mathcal{S}$. Consider the family of semi-norms

$$\|f\|_{(N,\alpha)} = \sup_{\mathbb{R}^n} (1+|x|)^N |\partial^\alpha f|$$

<u>Def.</u> The Schwartz space $\mathcal{S}$ is the vector space of $f \in C^\infty(\mathbb{R}^n)$ s.t.

$\|f\|_{(N,\alpha)} < \infty$, $\forall (N,\alpha) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n_{\geq 0}$,

endowed w/ the family of semi-norms $\|\cdot\|_{(N,\alpha)}$.

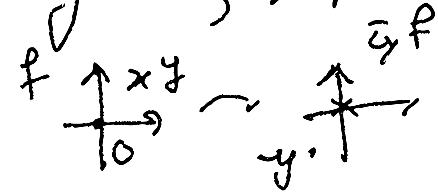
<u>Thm 1.</u> $\mathcal{S}$ is complete, i.e. a Frechet space.

<u>Pf.</u> See Folland Prop 8.2.

An important operation:

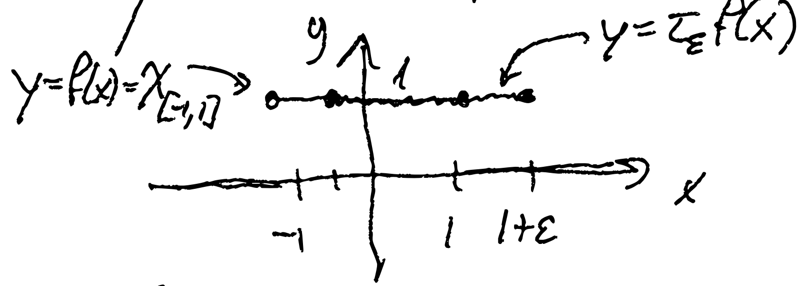
For $f: \mathbb{R}^n \to \mathbb{C}$ and $y \in \mathbb{R}^n$, let

$(\tau_y f)(x) := f(x-y)$.



<u>Prop 1.</u> τ_y is an isometry $L^p \to L^p$, and for $f \in L^p$, $\|f - \tau_y f\|_{L^p} \to 0$ as $y \to 0 \in \mathbb{R}^n$.

<u>Rem.</u> Clearly not true for L^∞:



$\|f - \tau_\varepsilon f\|_{L^\infty} = 1, \ \forall \varepsilon \neq 0$.

<u>Pf of Prop 1.</u> τ_y is clearly a linear operator $L^p \to L^p$, and since

$\tau_{-y} \circ \tau_y = \tau_y \circ \tau_{-y} = Id$, $\|\tau_y f\|_{L^p} = \|f\|_{L^p}$,

$\tau_y : L^p \to L^p$ is an isometry.

Next, pick $f \in L^p$ and $\varepsilon > 0$.

WTS, $\|f - \tau_y f\|_{L^p} \to 0$ as $y \to 0 \in \mathbb{R}^n$.

We first recall that C_c is dense in L^p (Prop 7.9; Lebesgue meas., restr. to $\mathcal{B}_{\mathbb{R}^n}$ is Radon) $\Rightarrow \exists\, \varphi \in C_c$ st. $\|f - \varphi\|_{L^p} = \|\tau_y f - \tau_y \varphi\|_{L^p} < \varepsilon$. Thus, $\|\cdot\| = \|\cdot\|_{L^p}$,

$$\|f - \tau_y f\| \le \|f - \varphi\| + \|\varphi - \tau_y \varphi\| + \|\tau_y \varphi - \tau_y f\|$$

$$< \|\varphi - \tau_y \varphi\| + 2\varepsilon$$

We conclude it suffices to prove $\|\varphi - \tau_y \varphi\|_{L^p} \to 0$, $\varphi \in C_c$, as $y \to 0 \in \mathbb{R}^n$.

Since φ has cpct supp $\Rightarrow$ φ is unif. cont., so $\exists\ \delta>0$ s.t. $|y|\le\delta \Rightarrow$ $\|\varphi-\tau_y\varphi\|_u<\varepsilon$. Let K_δ denote the cpct set $\overline{\{x : x-y\in \operatorname{supp}\varphi, |y|\le\delta\}}$

$$\|\varphi-\tau_y\varphi\|_{L^p}^p=\int_{\mathbb{R}^n}|\varphi-\tau_y\varphi|^p dx=\int_{K_\delta}|\varphi-\tau_y\varphi|^p dx$$

$$\le \|\varphi-\tau_y\varphi\|_u \cdot m(K_\delta) < m(K_\delta)\cdot\varepsilon.$$

$\Rightarrow \|\varphi-\tau_y\varphi\|_{L^p}\to 0$ as $y\to 0\in\mathbb{R}^n$

for $\varphi\in C_c$. $\blacksquare$

Convolutions.

For $f, g : \mathbb{R}^n \to \mathbb{C}$ meas., the convolution $f*g$ is defined by

$$(f*g)(x) = \int f(x-y)g(y)\,dy$$

$$\left(= \int (\tau_y f)(x) g(y)\,dy \right).$$

whenever this is defined, either $\forall y$ or a.e..

Basic Props (see Prop 8.6)

(a) $f*g = g*f$ (commutative)

(b) $(f*g)*h = f*(g*h)$ (associative)

(c) $\tau_z(f*g) = (\tau_z f)*g = f*(\tau_z g)$, $z \in \mathbb{R}^n$

(d) $A = \{x+y : x \in \operatorname{supp} f, y \in \operatorname{supp} g\}$

$\Rightarrow \operatorname{supp} f*g \subseteq A$.

We have

<u>Young's Inequality</u>. Let $1 \le p, q, r \le \infty$ be s.t. $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. If $f \in L^p$, $g \in L^q$, then $f*g \in L^r$ and

$$\|f*g\|_{L^r} \le \|f\|_p \|g\|_q .$$

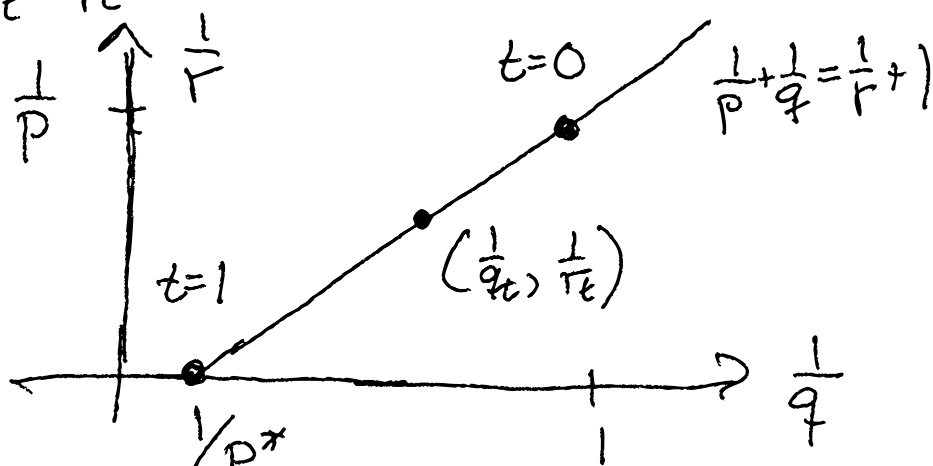
To prove this, we shall establish the "end point" estimates for fixed p, $(q,r) = (1,p)$ and $(q,r) = (q,\infty)$. The general result then follows from <u>Riesz-Thorin's Interpolation Thm</u>: Let $p_0, p_1, r_0, r_1 \in [1,\infty]$ and define

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{r_t} = \frac{1-t}{r_0} + \frac{t}{r_1}$$

If $T : L^{p_0} + L^{p_1} \to L^{r_0} + L^{r_1}$ (linear) s.t. $\|Tf\|_{L^{r_0}} \le C_0 \|f\|_{L^{p_0}}$ and $\|Tf\|_{L^{r_1}} \le C_1 \|f\|_{L^{p_1}}$, then

$$\|Tf\|_{L^{r_t}} \le C_0^{1-t} C_1^{t} \|f\|_{L^{p_t}}, \quad t \in (0,1).$$

To prove Y-I, we fix $p \in [1,\infty]$, $g \in L^p$, and let $Tf = f * g$. Let $p_0 = p_1 = p$, $r_0 = p$ and $r_1 = \infty$. If we let $q_0 = 1$, $q_1 = p^*$ $\left(\frac{1}{p} + \frac{1}{p^*} = 1\right)$ then w/ $\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$ we have: $\frac{1}{p} + \frac{1}{q_t} = \frac{1}{r_t} + 1$ for $t \in [0,1]$. Thus,



Thus, YI follows from RT if we establish the endpoint estimates

- $\|f*g\|_{L^p} \le \|g\|_{L^p} \cdot \|f\|_{L^1}$ $(t=0)$

- $\|f*\|_{L^\infty} \le \|g\|_{L^p} \|f\|_{L^{p^*}}$ $(t=1)$

The 1^{st} • is also known as (basic) YI:

<u>Young's Inequality (Basic)</u>. $f \in L^1$, $g \in L^p$, $1 \le p \le \infty \Rightarrow (f*g)(y)$ exists a.e. $y \in \mathbb{R}^n$ and

$$\|f*g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}.$$

<u>Pf.</u> Thm 6.18 w/ $K(x,y) = f(x-y)$.

(Exercise w/ Hölder's inequality!) ∎

For 2^{nd}•, we shall prove something stronger:

<u>Prop 2.</u> Let $p \in [1,\infty]$, $q = p^*$, and $f \in L^p$, $g \in L^q$. Then, $f * g$ exist $\forall x$ and is uniformly continuous. Moreover,

$$\|f * g\|_u \le \|f\|_{L^p} \|g\|_{L^q}.$$

<u>Pf.</u> The existence and estimate on $\|f*g\|_u$ is just Hölder's Ineq:

$$|(f*g)(x)| \le \int |f(x-y)|\,|g(y)|\,dy \le$$

$$\|f(x-\cdot)\|_{L^p} \|g\|_{L^q} = \|f\|_{L^p} \|g\|_{L^q}$$

To prove unif. continuity, we also use Hölder:

$$|(f*g)(x-y) - (f*g)(x)| \le$$

$$\int |f(x-y-z) - f(x-z)|\,|g(z)|\,dz \le$$

$$\left(\int |f(x-y-z) - f(x-z)|^p dz\right)^{1/p} \|g\|_{L^q} =$$

$$\left(\int |f(z'-y) - f(z')|^p dz'\right)^{1/p} \|g\|_{L^q}$$

$$= \|\tau_y f - f\|_{L^p} \|g\|_{L^q} \Rightarrow$$

$$\|\tau_y(f*g) - f*g\|_u \le \|\tau_y f - f\|_{L^p} \|g\|_{L^q}$$

By Prop 1, $\|\tau_y(f*g) - f*g\|_u \to 0$ as $y \to 0 \in \mathbb{R}^n$, i.e. $f*g$ is unif. cont. $\boxtimes$